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# HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE TRANS-SASAKIAN MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. We study half lightlike submanifolds $M$ of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection subject to the conditions that the characteristic vector field $\zeta$ of $\bar{M}$ is identical with the indefinite trans-Sasakian structure vector field $\zeta$ of $\bar{M}$ and $\zeta$ is tangent to $M$. Under the same conditions, we also characterize half lightlike submanifolds of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_2)$ with a semi-symmetric metric connection.

## 1. Introduction

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a *semi-symmetric connection* if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(\bar{X}, \bar{Y}) = \theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}, \tag{1.1}$$

where $\theta$ is a 1-form associated with a smooth unit vector field $\zeta$, which is called the *characteristic vector field*, by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. Moreover, if this connection $\bar{\nabla}$ is a metric connection, *i.e.*, it satisfies $\bar{\nabla}\bar{g} = 0$, then $\bar{\nabla}$ is called a *semi-symmetric metric connection*. The notion of semi-symmetric metric connection on a Riemannian manifold was introduced by Yano [13]. In the followings, we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

*Remark* 1.1. Denote $\widetilde{\nabla}$ by the Levi-Civita connection of the manifold $(\bar{M}, \bar{g})$ with respect to the metric $\bar{g}$. It is well known that *a linear connection* $\bar{\nabla}$ *on* $\bar{M}$ *is a semi-symmetric metric connection if and only if it satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})\bar{X} - \bar{g}(\bar{X}, \bar{Y})\zeta. \tag{1.2}$$

A submanifold $M$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called *lightlike submanifold* if $Rad(TM) = TM \cap TM^{\perp} \neq \{0\}$, where $TM$ and $TM^{\perp}$ are the tangent and normal bundle of $M$, respectively. A codimension 2 lightlike submanifold $M$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called

(1) *half lightlike submanifold* if $rank\{Rad(TM)\} = 1$,
(2) *coisotropic submanifold* if $rank\{Rad(TM)\} = 2$.

Half lightlike submanifold was introduced by Duggal-Bejancu [4] and later, studied by Duggal-Jin [6]. It is a special case of $r$-lightlike submanifold [5]. Its geometry is more general than that of lightlike hypersurface or coisotropic submanifold of codimension 2. Much of its theory will be immediately generalized in a formal way to general $r$-lightlike submanifolds.

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The notion of a trans-Sasakian manifold, of type $(\alpha, \beta)$, was introduced by Oubina [12]. Sasakian, Kenmotsu and cosymplectic manifolds are three important kinds of trans-Sasakian manifold given respectively by

$$\alpha = \varepsilon,\ \beta = 0, \qquad \alpha = 0,\ \beta = \varepsilon, \qquad \alpha = \beta = 0, \qquad \varepsilon = \pm 1.$$

The generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ was introduced by Alegre *et. al.* [1]. Sasakian, Kenmotsu and cosymplectic space forms are three important kinds of generalized Sasakian space form given respectively by

$$f_1 = \tfrac{c+3}{4}, f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4}, f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4},$$

where $c$ is a constant J-sectional curvature of each space forms.

In this paper, we study half lightlike submanifolds $M$ of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection, in which the characteristic vector field $\zeta$ is identical with the structure vector field $\zeta$ of the indefinite trans-Sasakian structure $(J, \zeta, \theta, \bar{g})$ of $\bar{M}$ and $\zeta$ is tangent to $M$. In Section 4, we characterize half lightlike submanifolds equipped with recurrent or Lie recurrent structure tensor field $F$. In Section 5, we study half lightlike submanifolds of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a semi-symmetric metric connection.

## 2. Semi-symmetric metric connections

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an *indefinite almost contact metric manifold* if there exists a set $\{J, \zeta, \theta, \bar{g}\}$, where $J$ is a (1, 1)-type tensor field, $\zeta$ is a vector field and $\theta$ is a 1-form such that

$$(2.1)\quad J^2\bar{X} = -\bar{X} + \theta(\bar{X})\zeta,\ \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \epsilon\theta(\bar{X})\theta(\bar{Y}),\ \theta(\zeta) = 1,$$
$$J\zeta = 0, \quad \theta \circ J = 0, \quad \theta(\bar{X}) = \epsilon\bar{g}(\bar{X}, \zeta), \quad \bar{g}(J\bar{X}, \bar{Y}) = -\bar{g}(\bar{X}, J\bar{Y}),$$

where $\epsilon = 1$ or $-1$ according as $\zeta$ is spacelike or timelike, respectively.

In the entire discussion of this article, we shall assume that the structure vector field $\zeta$ is a spacelike one, *i.e.*, $\epsilon = 1$, without loss of generality.

**Definition.** An indefinite almost contact metric manifold $(\bar{M}, \bar{g})$ is said to be an *indefinite trans-Sasakian manifold* [12] if there exist a Levi-Civita connection $\widetilde{\nabla}$ with respect to $\bar{g}$ and two smooth functions $\alpha$ and $\beta$ such that

$$(\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\}.$$

$\{J, \zeta, \theta, \bar{g}\}$ is called an *indefinite trans-Sasakian structure of type* $(\alpha, \beta)$.

Using (1.2), (2.1) and the facts that $J\zeta = 0$ and $\theta \circ J = 0$, we see that

$$(2.2)\qquad (\bar{\nabla}_{\bar{X}}J)\bar{Y} = \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + (\beta + 1)\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\}.$$

Replacing $\bar{Y}$ by $\zeta$ to (2.2) and using $J\zeta = 0$ and $\theta(\bar{\nabla}_{\bar{X}}\zeta) = 0$, we obtain

$$(2.3)\qquad \bar{\nabla}_{\bar{X}}\zeta = -\alpha J\bar{X} + (\beta + 1)\{\bar{X} - \theta(\bar{X})\zeta\}.$$

Let $(M, g)$ be a half lightlike submanifold of $\bar{M}$. As $rank(Rad(TM)) = 1$, there exist two complementary non-degenerate vector bundles $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in $TM$ and $TM^{\perp}$, respectively, which are called *screen distribution* and *co-screen distribution* of $M$, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$. Also denote by $(2.1)_i$ the $i$-th equation of the seven equations in (2.1). We use the same notations for any others. Choose $L$ as a unit spacelike vector field on $S(TM^{\perp})$, no loss of generality. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$. Certainly, $Rad(TM)$ and $S(TM^{\perp})$ are vector subbundles of $S(TM)^{\perp}$. As the co-screen distribution $S(TM^{\perp})$ is non-degenerate, we have

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp},$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section $\xi$ of $Rad(TM)$, there exists a uniquely defined lightlike vector bundle $ltr(TM)$ and a null vector field $N$ of $ltr(TM)$ satisfying

$$\bar{g}(\xi, N) = 1,\ \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0,\ \forall X \in \Gamma(S(TM)).$$

We call $N$, $ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$ the *null transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of $M$ with respect to $S(TM)$, respectively [6]. $T\bar{M}$ is decomposed as

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

Denote by $X$, $Y$ and $Z$ the smooth vector fields on $M$, unless otherwise specified. Let $P$ be the projection morphism of $TM$ on $S(TM)$. The local Gauss and Weingarten formulae of $M$ and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.5}$$

$$\bar{\nabla}_X L = -A_L X + \phi(X)N; \tag{2.6}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.7}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.8}$$

respectively, where $\nabla$ and $\nabla^*$ are linear connections on $M$ and $S(TM)$, respectively, $B$ and $D$ are the *local second fundamental forms* of $M$, $C$ is the *local second fundamental form* on $S(TM)$. $A_N$, $A_L$ and $A^*_\xi$ are the *shape operators* and $\tau$, $\rho$ and $\phi$ are 1-forms on $M$.

Using (1.1) and (2.4), we see that $B$ and $D$ are symmetric, and

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \tag{2.9}$$

$$T(X, Y) = \theta(Y)X - \theta(X)Y, \tag{2.10}$$

where $T$ is the torsion tensor with respect to $\nabla$ and $\eta$ is a 1-form such that

$$\eta(X) = \bar{g}(X, N).$$

From the facts that $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X, Y) = \bar{g}(\bar{\nabla}_X Y, L)$, we show that $B$ and $D$ are independent of the choice of the screen distribution $S(TM)$. Applying $\bar{\nabla}_X$ to $\bar{g}(\xi, \xi) = 0$, $\bar{g}(\xi, L) = 0$, $\bar{g}(N, N) = 0$ and $\bar{g}(N, L) = 0$ by turns and using (1.1) and (2.4) $\sim$ (2.6), we obtain

$$B(X, \xi) = 0, \qquad D(X, \xi) = -\phi(X), \tag{2.11}$$

$$\bar{g}(A_N X, N) = 0, \qquad \bar{g}(A_L X, N) = \rho(X). \tag{2.12}$$

The second fundamental forms are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \tag{2.13}$$

$$D(X,Y) = g(A_{_L} X, Y) - \phi(X)\eta(Y), \tag{2.14}$$

$$C(X,PY) = g(A_{_N} X, PY). \tag{2.15}$$

From (2.8), $(2.11)_1$ and (2.13), $A^*_\xi$ is $S(TM)$-value and satisfies

$$A^*_\xi \xi = 0. \tag{2.16}$$

**Definition.** A half lightlike submanifold $M$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called *statical* [9] if $\bar{\nabla}_X L \in \Gamma(S(TM))$ for any $X \in \Gamma(TM)$.

From (2.6) and $(2.12)_2$, we show that the above definition is equivalent to the conditions: $\phi = 0$ and $\rho = 0$. $\phi = 0$ is equivalent to the conception: $M$ is *irrotational, i.e.*, $\bar{\nabla}_X \xi \in \Gamma(TM)$ [11]. $\rho = 0$ is equivalent to the conception: $M$ is *solenoidal, i.e.*, $A_{_L} X \in \Gamma(S(TM))$ [10].

## 3. Structure equations of $M$

It is known [7] that, for a half lightlike submanifold $M$ of an indefinite almost contact manifold $\bar{M}$, there exists a screen distribution $S(TM)$ such that the vector bundles $J(Rad(TM)), J(ltr(TM))$ and $J(S(TM^\perp))$ are subbundles of $S(TM)$ with mutually trivial intersections, of rank 1. Thus

$$J(Rad(TM)) \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\perp))$$

is a subbundle of $S(TM)$, of rank 3. Now we shall assume that the structure vector field $\zeta$ of $\bar{M}$ is tangent to $M$. Călin [2] proved that if $\zeta$ is tangent to $M$, then it belongs to $S(TM)$. Then there exist two non-degenerate almost complex distributions $H_o$ and $H$ on $M$ with respect to $J$ such that

$$S(TM) = J(Rad(TM)) \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\perp) \oplus_{orth} H_o,$$
$$H = \{Rad(TM) \oplus_{orth} J(Rad(TM))\} \oplus_{orth} H_o.$$

In this case, the decomposition form of $TM$ is reduced to

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\perp)). \tag{3.1}$$

Consider two null and one spacelike vector fields $\{U, V\}$ and $W$ such that

$$U = -JN, \quad V = -J\xi, \qquad W = -JL. \tag{3.2}$$

Denote by $S$ the projection morphism of $TM$ on $H$ with respect to the decomposition (3.1). Any vector field $X$ on $M$ is expressed as follows

$$X = SX + u(X)U + w(X)W,$$

where $u$, $v$ and $w$ are 1-forms locally defined on $M$ by

$$u(X) = g(X, V), \quad v(X) = g(X, U), \quad w(X) = g(X, W). \tag{3.3}$$

Using (3.2), the action $JX$ of $X$ by $J$ is expressed as follow:

$$JX = FX + u(X)N + w(X)L, \tag{3.4}$$

where $F$ is a tensor field of type $(1,1)$ globally defined on $M$ by $F = J \circ S$. Applying $J$ to (3.4) and using (2.1) and (3.2), we have

$$F^2 X = -X + u(X)U + w(X)W + \theta(X)\zeta. \tag{3.5}$$

We say that $F$ is the *structure tensor field* of $M$.

Substituting (3.4) into (2.3) and using (2.4), we have

$$\nabla_X \zeta = -\alpha FX + (\beta+1)\{X - \theta(X)\zeta\}, \tag{3.6}$$

$$B(X,\zeta) = -\alpha u(X), \qquad D(X,\zeta) = -\alpha w(X). \tag{3.7}$$

Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta, N) = 0$ and using (2.3), (2.5) and (2.15), we have

$$C(X,\zeta) = -\alpha v(X) + (\beta+1)\eta(X). \tag{3.8}$$

Applying $\bar{\nabla}_X$ to (3.2) ∼ (3.4) and using (2.2), (2.4)∼(2.6), (2.8), (2.11), (2.13) ∼ (2.15) and (3.2) ∼ (3.4), we have

$$B(X,U) = C(X,V), \; D(X,U) = C(X,W), \; D(X,V) = B(X,W), \tag{3.9}$$

$$\begin{aligned} \nabla_X U = {} & F(A_N X) + \tau(X)U + \rho(X)W \\ & - \{\alpha\eta(X) + (\beta+1)v(X)\}\zeta, \end{aligned} \tag{3.10}$$

$$\nabla_X V = F(A^*_\xi X) - \tau(X)V - \phi(X)W - (\beta+1)u(X)\zeta, \tag{3.11}$$

$$\nabla_X W = F(A_L X) + \phi(X)U - (\beta+1)w(X)\zeta, \tag{3.12}$$

$$\begin{aligned} (\nabla_X F)Y = {} & u(Y)A_N X + w(Y)A_L X - B(X,Y)U - D(X,Y)W \\ & + \alpha\{g(X,Y)\zeta - \theta(Y)X\} \\ & + (\beta+1)\{\bar{g}(JX,Y)\zeta - \theta(Y)FX\}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} (\nabla_X u)(Y) = {} & -u(Y)\tau(X) - w(Y)\phi(X) - B(X,FY) \\ & - (\beta+1)\theta(Y)u(X), \end{aligned} \tag{3.14}$$

$$\begin{aligned} (\nabla_X v)(Y) = {} & v(Y)\tau(X) + w(Y)\rho(X) - g(A_{N_i}X, FY) \\ & - \{\alpha\eta(X) + (\beta+1)v(X)\}\theta(Y). \end{aligned} \tag{3.15}$$

**Theorem 3.1.** *Let $M$ be a half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection. If either $U$ or $V$ is parallel with respect to the induced connection $\nabla$, then $\tau = 0$ and $\bar{M}$ is an indefinite Kenmotsu manifold, i.e., $\alpha = 0$ and $\beta = -1$.*

*Proof.* (1) If $U$ is parallel with respect to the connection $\nabla$, then, taking the scalar product with $\zeta$, $V$ and $W$ to (3.10) by turns, we get

$$\alpha = 0, \quad \beta = -1\,; \qquad \tau = 0\,; \qquad \rho = 0. \tag{3.16}$$

As $\alpha = 0$ and $\beta = -1$, $\bar{M}$ is an indefinite Kenmotsu manifold. Applying $F$ to (3.10): $F(A_N X) = 0$ and using (2.15), (3.5) and (3.8), we obtain

$$A_N X = u(A_N X)U + w(A_N X)W. \tag{3.17}$$

(2) If $V$ is parallel with respect to $\nabla$, then, taking the scalar product with $\zeta$, $U$ and $W$ to (3.11) by turns, we have $\beta = -1$, $\tau = 0$ and $\phi = 0$. Applying $F$ to (3.11): $F(A^*_\xi X) = 0$ and using (3.5) and $(3.7)_1$, we obtain

$$A^*_\xi X = -\alpha u(X)\zeta + u(A^*_\xi X)U + w(A^*_\xi X)W.$$

Taking the scalar product with $U$ to this equation, we have

$$B(X,U) = 0. \tag{3.18}$$

Replacing $X$ by $\zeta$ to (3.18) and using $(3.7)_1$, we have $\alpha = 0$. Thus

$$\alpha = 0, \qquad \beta = -1, \qquad \tau = 0, \qquad \phi = 0. \tag{3.19}$$

As $\alpha = 0$ and $\beta = -1$, $\bar{M}$ is an indefinite Kenmotsu manifold. □

## 4. Recurrent and Lie recurrent submanifolds

**Definition.** The structure tensor field $F$ of $M$ is said to be *recurrent* [8] if there exists a 1-form $\varpi$ on $M$ such that

$$(\nabla_X F)Y = \varpi(X)FY.$$

A half lightlike submanifold $M$ of an indefinite trans-Sasakian manifold $\bar{M}$ is called *recurrent* if it admits a recurrent structure tensor field $F$.

**Theorem 4.1.** *Let $M$ be a recurrent half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with a semi-symmetric metric connection. Then the following five statements are satisfied*

(1) *$\bar{M}$ is an indefinite Kenmotsu manifold, i.e., $\alpha = 0$ and $\beta = -1$,*
(2) *$F$ is parallel with respect to the induced connection $\nabla$ on $M$,*
(3) *$M$ is statical, i.e., $\phi = 0$ and $\rho = 0$,*
(4) *$H$, $J(S(TM^{\perp}))$ and $J(ltr(TM))$ are parallel distributions on $M$,*
(5) *$M$ is locally a product manifold $\mathcal{C}_U \times \mathcal{C}_W \times M^{\sharp}$, where $\mathcal{C}_U$ is a null curve tangent to $J(ltr(TM))$, $\mathcal{C}_W$ is a spacelike curve tangent to $J(S(TM^{\perp}))$, and $M^{\sharp}$ is a leaf of the distribution $H$.*

*Proof.* (1) From the above definition and (3.13), we get

$$\begin{aligned}\varpi&(X)FY \\ &= u(Y)A_N X + w(Y)A_L X - B(X,Y)U - D(X,Y)W \\ &+ \alpha\{g(X,Y)\zeta - \theta(Y)X\} + (\beta+1)\{\bar{g}(JX,Y)\zeta - \theta(Y)FX\}.\end{aligned} \tag{4.1}$$

Replacing $Y$ by $\zeta$ to (4.1) and using (3.5) and $(3.7)_{1,2}$, we obtain

$$\alpha F^2 X = (\beta+1)FX.$$

Taking $X = \xi$ to this equation and using the fact that $F\xi = -V$, we have

$$\alpha\xi = (\beta+1)V.$$

Taking the scalar product with $N$ and $U$ to this equation by turns, we get

$$\alpha = 0, \qquad \beta = -1. \tag{4.2}$$

Therefore, $\bar{M}$ is an indefinite Kenmotsu manifold.

(2) Replacing $Y$ by $\xi$ to (4.1) and using (2.11) and (4.2), we obtain

$$\varpi(X)V + \phi(X)W = 0.$$

Taking the scalar product with $U$ and $W$ to this equation, we have

$$\varpi = 0, \qquad \phi = 0. \tag{4.3}$$

As $\varpi = 0$, $F$ is parallel with respect to the induced connection $\nabla$ on $M$.

(3) As $\phi = 0$, $M$ is irrotational. Taking the scalar product with $N$ to (4.1) and using (2.12), (4.2) and (4.3), we get $w(Y)\rho(X) = 0$. It follow that $\rho = 0$. Thus $M$ is solenoidal. Therefore, $M$ is statical.

(4) Taking the scalar product with $V$ and $W$ to (4.1) by turns, we obtain

$$B(X,Y) = u(Y)u(A_N X) + w(Y)u(A_L X),$$
$$D(X,Y) = u(Y)w(A_N X) + w(Y)w(A_L X).$$

Taking $Y = V$ and $Y = FZ_o$, $Z_o \in \Gamma(H_o)$ to these two equations by turns, and using the fact that $u(FZ_o) = w(FZ_o) = 0$, we have

$$B(X, V) = 0, \quad B(X, FZ_o) = 0; \quad D(X, V) = 0, \quad D(X, FZ_o) = 0. \tag{4.4}$$

In general, by using (2.8), (2.13), (2.14), (3.9), (3.11), (3.12), we derive

$$\begin{aligned} &g(\nabla_X \xi, V) = -B(X, V), && g(\nabla_X \xi, W) = -D(X, V), \\ &g(\nabla_X V, V) = 0, && g(\nabla_X V, W) = -\phi(X), \\ &g(\nabla_X Z_o, V) = B(X, FZ_o), && g(\nabla_X Z_o, W) = D(X, FZ_o). \end{aligned}$$

From these equations, $(4.3)_2$ and (4.4), we see that

$$\nabla_X Y \in \Gamma(H), \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(H).$$

It follows that $H$ is a parallel distribution on $M$.

Taking $Y = U$ and $Y = W$ to (4.1) by turns and using (4.2), we have

$$\begin{aligned} A_N X &= B(X, U)U + D(X, U)W, \\ A_L X &= B(X, W)U + D(X, W)W. \end{aligned}$$

Applying $F$ to these equations and using the fact $FU = FW = 0$, we get

$$F(A_N X) = 0, \qquad F(A_L X) = 0.$$

Using this result, (4.2) and (4.3), Eq.s (3.10) and (3.12) are reduced to

$$\nabla_X U = \tau(X)U, \qquad \nabla_X W = 0. \tag{4.5}$$

Thus $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are also parallel distributions on $M$

(5) As $H$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are parallel distributions satisfying (3.1), by the de Rham's decomposition theorem [3], $M$ is locally a product manifold $\mathcal{C}_U \times \mathcal{C}_W \times M^{\sharp}$, where $\mathcal{C}_U$ is a null curve tangent to $J(ltr(TM))$, $\mathcal{C}_W$ is a spacelike curve tangent to $J(S(TM^{\perp}))$ and $M^{\sharp}$ is a leaf of $H$. □

**Definition.** The structure tensor field $F$ of $M$ is said to be *Lie recurrent* [8] if there exists a 1-form $\vartheta$ on $M$ such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on $M$ with respect to $X$, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y].$$

The structure tensor field $F$ is called *Lie parallel* if $\mathcal{L}_X F = 0$. A half lightlike submanifold $M$ of an indefinite Kaehler manifold $\bar{M}$ is called *Lie recurrent* if it admits a Lie recurrent structure tensor field $F$.

**Theorem 4.2.** *Let $M$ be a Lie recurrent half lightlike submanifold of an indefinite trans-Sasakian manifold with a semi-symmetric metric connection. Then the following four statements are satisfied*

(1) $\alpha = 0$ *and $\bar{M}$ is an indefinite $\beta$-Kenmotsu manifold,*
(2) *$F$ is Lie parallel,*
(3) *$\tau$ and $\rho$ satisfy $\tau \circ F = \rho \circ F = 0$. Moreover, $\tau = -\beta\theta$ on $TM$,*
(4) *the shape operator $A^*_\xi$ satisfies $A^*_\xi U = 0$ and $A^*_\xi V = 0$.*

*Proof.* (1) Using the above definition, (2.10) and the fact $\theta(FY)=0$, we get

$$\vartheta(X)FY = (\nabla_X F)Y - \nabla_{FY}X + F\nabla_Y X + \theta(Y)FX.$$

Substituting (3.11) into the last equation, we obtain

$$\begin{aligned}\vartheta(X)FY = & -\nabla_{FY}X + F\nabla_Y X + u(Y)A_{_N}X + w(Y)A_{_L}X \\ & - B(X,Y)U - D(X,Y)W - \beta\theta(Y)FX \\ & + \alpha\{g(X,Y)\zeta - \theta(Y)X\} + (\beta+1)\bar{g}(JX,Y)\zeta.\end{aligned} \tag{4.6}$$

Taking $Y=\xi$ to (4.6) and using (2.11), we have

$$-\vartheta(X)V = \nabla_V X + F\nabla_\xi X + \phi(X)W + (\beta+1)u(X)\zeta. \tag{4.7}$$

Taking the scalar product with $V$, $W$ and $\zeta$ to (4.7) by turns, we have

$$u(\nabla_V X) = 0, \quad w(\nabla_V X) = -\phi(X), \quad \theta(\nabla_V X) = -(\beta+1)u(X). \tag{4.8}$$

Replacing $Y$ by $V$ to (4.6) and using the fact that $\theta(V)=0$, we have

$$\vartheta(X)\xi = -\nabla_\xi X + F\nabla_V X - B(X,V)U - D(X,V)W + \alpha u(X)\zeta. \tag{4.9}$$

Taking the scalar product with $\zeta$ to (4.9) such that $X=U$ and using (3.10), we obtain $\alpha=0$. Thus $\bar{M}$ is an indefinite $\beta$-Kenmotsu manifold

(2) Applying $F$ to (4.9) and using (3.5) and (4.8), we obtain

$$\vartheta(X)V = \nabla_V X + F\nabla_\xi X + \phi(X)U + (\beta+1)u(X)\zeta.$$

Comparing this equation with (4.7), we get $\vartheta=0$. Thus $F$ is Lie parallel.

(3) Taking the scalar product with $N$ to (4.7) such that $X=W$ and using $(2.12)_2$, (2.14) and (3.12), we get $D(U,V)=\rho(\xi)$. Also, taking the scalar product with $W$ to (4.9) such that $X=U$ and using (3.10), we have $D(U,V)=-\rho(\xi)$. Thus $\rho(\xi)=0$ and $D(U,V)=0$.

Taking $X=U$ to $(4.8)_2$ and using (3.10), we get $\rho(V)=-\phi(U)$. Also, taking the product with $U$ to (4.7) such that $X=W$ and using $(2.11)_2$, $(2.13)_2$ and (3.12), we get $\rho(V)=\phi(U)$. Thus $\rho(V)=0$ and $\phi(U)=0$.

Replacing $X$ by $W$ to $(4.8)_1$ and using (3.12), we get $\phi(V)=0$.

Taking the scalar product with $W$ to (4.7) such that $X=\xi$ and using (2.8) and (2.13), we get $B(V,W)=\phi(\xi)$. Also, taking the scalar product with $V$ to (4.8) such that $X=W$ and using (3.12), we have $B(V,W)=-\phi(\xi)$. Thus $\phi(\xi)=0$ and $B(V,W)=0$.

Summarizing the above results, we obtain

$$\begin{aligned}&\rho(\xi)=0, \quad \rho(V)=0, \quad \phi(U)=0, \quad \phi(V)=0, \quad \phi(\xi)=0,\\ &D(U,V)=B(U,W)=0, \qquad B(V,W)=D(V,V)=0.\end{aligned} \tag{4.10}$$

Taking the scalar product with $N$ to (4.6) and using (2.12), we have

$$-\bar{g}(\nabla_{FY}X,N) + \bar{g}(\nabla_Y X,U) = \beta\theta(Y)v(X) - w(Y)\rho(X). \tag{4.11}$$

Replacing $X$ by $\xi$ to this and using (2.8), (2.13) and $(4.10)_1$, we obtain

$$B(X,U) = \tau(FX). \tag{4.12}$$

Taking $X=U$ to (4.12) and using (3.8) and the fact that $FU=0$, we get

$$C(U,V) = B(U,U) = 0. \tag{4.13}$$

Taking $X = U$ to (4.6) and using (2.14), (2.15), (3.3), (3.5), (3.8), $(3.9)_{1,2}$, (3.10) and the facts that $\alpha = 0$ and $FU = FW = F\zeta = 0$, we obtain

$$\begin{aligned} u(Y)A_{_N}U + w(Y)A_{_L}U - F(A_{_N}FY) - \tau(FY)U \\ - \rho(FY)W - A_{_N}Y + (\beta+1)\eta(Y)\zeta = 0. \end{aligned} \tag{4.14}$$

Taking the scalar product with $V$ and using $(4.10)_6$ and (4.13), we get

$$B(X,U) = -\tau(FX).$$

Comparing this equation with (4.12), we obtain $\tau(FX) = 0$.

Replacing $X$ by $V$ to (4.11) and using (3.11) and $(4.10)_2$, we have

$$B(FY,U) + \beta\theta(Y) = -\tau(Y).$$

Taking $Y = U$, $Y = W$ and $Y = \zeta$ and using $FU = FW = F\zeta = 0$, we get

$$\tau(U) = 0, \qquad \tau(W) = 0, \qquad \tau(\zeta) = -\beta. \tag{4.15}$$

Taking $X = FY$ to $\tau(FX) = 0$ and using (3.5) and (4.15), we get $\tau = -\beta\theta$.

Replacing $Y$ by $W$ to (4.14), we obtain $A_{_N}W = A_{_L}U$. Taking the scalar product with $U$ to this result and using (2.14), (2.15) and $(3.9)_2$, we have

$$C(W,U) = D(U,U) = C(U,W). \tag{4.16}$$

Taking the scalar product with $W$ to (4.14) and using (2.14), we have

$$-\rho(FY) = C(Y,W) - u(Y)C(U,W) - w(Y)D(U,W).$$

Taking the scalar product with $U$ to (4.6) such that $X = W$ and using $(2.12)_2$, (2.14), $(3.9)_2$, (3.12) and (4.16), we obtain

$$\rho(FY) = C(Y,W) - u(Y)C(U,W) - w(Y)D(U,W).$$

Comparing the last two equations, we obtain $\rho(FY) = 0$.

(4) As $\tau(FX) = 0$, from (2.11) and (4.12), we have $g(A^*_\xi U, X) = 0$, As $S(TM)$ is non-degenerate, we get $A^*_\xi U = 0$. Replacing $X$ by $\xi$ to (4.7) and using (2.8), (2.16), $(4.10)_5$ and $\tau(FX) = 0$, we obtain $A^*_\xi V = 0$. □

## 5. Generalized Sasakian space forms

**Definition.** An indefinite trans-Sasakian manifold $(\bar{M}, J, \zeta, \theta, \bar{g})$ is called an *indefinite generalized Sasakian space form* $\bar{M}(f_1, f_2, f_3)$ [1] if there exist three smooth functions $f_1$, $f_2$ and $f_3$ on $\bar{M}$ such that

$$\begin{aligned} \widetilde{R}(X,Y)Z \,=\, & f_1\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y}\} \\ & + f_2\{\bar{g}(\bar{X},J\bar{Z})J\bar{Y} - \bar{g}(\bar{Y},J\bar{Z})J\bar{X} + 2\bar{g}(\bar{X},J\bar{Y})J\bar{Z}\} \\ & + f_3\{\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X} \\ & \qquad + \bar{g}(\bar{X},\bar{Z})\theta(\bar{Y})\zeta - \bar{g}(\bar{Y},\bar{Z})\theta(\bar{X})\zeta\}, \end{aligned} \tag{5.1}$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\bar{M}$.

Let $\bar{R}$ be the curvature tensor of the semi-symmetric metric connection $\bar{\nabla}$ on $\bar{M}$. By directed calculations from (1.1) and (1.2), we see that

$$\begin{aligned} \bar{R}(\bar{X},\bar{Y})\bar{Z} = \widetilde{R}(\bar{X},\bar{Y})\bar{Z} + \bar{g}(\bar{X},\bar{Z})\bar{\nabla}_{\bar{Y}}\zeta - \bar{g}(\bar{Y},\bar{Z})\bar{\nabla}_{\bar{X}}\zeta \\ + \{(\bar{\nabla}_{\bar{X}}\theta)(\bar{Z}) - \bar{g}(\bar{X},\bar{Z})\}\bar{Y} - \{(\bar{\nabla}_{\bar{Y}}\theta)(\bar{Z}) - \bar{g}(\bar{Y},\bar{Z})\}\bar{X}. \end{aligned} \tag{5.2}$$

Denote by $R$ and $R^*$ the curvature tensors of the induced linear connections $\nabla$ and $\nabla^*$ on $M$ and $S(TM)$ respectively. Using the Gauss-Weingarten formulae, we obtain Gauss equations for $M$ and $S(TM)$ respectively:

$$\begin{aligned}(5.3)\quad \bar{R}(X,Y)Z &= R(X,Y)Z + B(X,Z)A_{N}Y - B(Y,Z)A_{N}X\\ &\quad + D(X,Z)A_{L}Y - D(Y,Z)A_{L}X\\ &\quad + \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)\\ &\qquad + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\\ &\qquad + \phi(X)D(Y,Z) - \phi(Y)D(X,Z) + B(T(X,Y),Z)\}N,\\ &\quad + \{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z)\\ &\qquad + \rho(X)B(Y,Z) - \rho(Y)B(X,Z) + D(T(X,Y),Z)\}L,\end{aligned}$$

$$\begin{aligned}(5.4)\quad R(X,Y)PZ &= R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X\\ &\quad + \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) - \tau(X)C(Y,PZ)\\ &\qquad + \tau(Y)C(X,PZ) + C(T(X,Y),PZ)\}\xi.\end{aligned}$$

Taking the scalar product with $\xi$ and $N$ to (5.2) by turns and then, substituting (5.1) and (5.3) and using (2.3), (2.10), (2.12) and (5.4), we get

$$\begin{aligned}(5.5)\qquad &(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)\\ &+ \{\tau(X) - \theta(X)\}B(Y,Z) - \{\tau(Y) - \theta(Y)\}B(X,Z)\\ &+ \phi(X)D(Y,Z) - \phi(Y)D(X,Z)\\ &+ \alpha\{u(Y)g(X,Z) - u(X)g(Y,Z)\}\\ &= f_2\{u(Y)\bar{g}(X,JZ) - u(X)\bar{g}(Y,JZ) + 2u(Z)\bar{g}(X,JY)\},\end{aligned}$$

$$\begin{aligned}(5.6)\qquad &(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ)\\ &- \{\tau(X) + \theta(X)\}C(Y,PZ) + \{\tau(Y) + \theta(Y)\}C(X,PZ)\\ &- \rho(X)D(Y,PZ) + \rho(Y)D(X,PZ)\\ &- \{(\bar{\nabla}_X\theta)(PZ) + \beta g(X,PZ)\}\eta(Y)\\ &+ \{(\bar{\nabla}_Y\theta)(PZ) + \beta g(Y,PZ)\}\eta(X)\\ &+ \alpha\{v(Y)g(X,PZ) - v(X)g(Y,PZ)\}\\ &= f_1\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y)\}\\ &+ f_2\{v(Y)\bar{g}(X,JPZ) - v(X)\bar{g}(Y,JPZ) + 2v(PZ)\bar{g}(X,JY)\}\\ &+ f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ).\end{aligned}$$

**Theorem 5.1.** *Let $M$ be a half lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a semi-symmetric metric connection. Then the functions $\alpha$, $\beta$, $f_1$, $f_2$ and $f_3$ satisfy*

(1) *$\alpha$ is a constant on $M$,*
(2) *$\alpha\beta = 0$, and*
(3) *$f_1 - f_2 = \alpha^2 - \beta^2$, $\qquad f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta$.*

*Proof.* Applying $\bar{\nabla}_X$ to $\theta(V) = 0$ and using (2.4) and (3.11), we obtain

$$(5.7)\qquad (\bar{\nabla}_X\theta)(V) = (\beta + 1)u(X).$$

Applying $\nabla_X$ to $(3.9)_1$: $B(Y,U) = C(Y,V)$ and using (2.13), (2.15), $(3.7)_1$, (3.8), $(3.9)_{1,2,3}$, (3.10) and (3.11), we get

$$\begin{aligned}(\nabla_X B)(Y,U) = (\nabla_X C)(Y,V) &- 2\tau(X)C(Y,V)\\ &- \phi(X)D(Y,U) - \rho(X)D(Y,V)\\ &- \alpha(\beta+1)\{u(Y)v(X) - u(X)v(Y)\}\\ &- \alpha^2 u(Y)\eta(X) - (\beta+1)^2 u(X)\eta(Y)\\ &- g(A^*_\xi X, F(A_N Y)) - g(A^*_\xi Y, F(A_N X)).\end{aligned}$$

Substituting this equation and $(3.9)_1$ into (5.5) with $Z = U$, we get

$$\begin{aligned}&(\nabla_X C)(Y,V) - (\nabla_Y C)(X,V)\\ &- \{\tau(X)+\theta(X)\}C(Y,V) + \{\tau(Y)+\theta(Y)\}C(X,V)\\ &- \rho(X)D(Y,V) + \rho(Y)D(X,V)\\ &- \alpha(2\beta+1)\{u(Y)v(X) - u(X)v(Y)\}\\ &- \{\alpha^2 - (\beta+1)^2\}\{u(Y)\eta(X) - u(X)\eta(Y)\}\\ &= f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X,JY)\}.\end{aligned}$$

Comparing this equation with (5.6) such that $PZ = V$, we obtain

$$\begin{aligned}&\{f_1 - f_2 - \alpha^2 + \beta^2\}\{u(Y)\eta(X) - u(X)\eta(Y)\}\\ &= 2\alpha\beta\{u(Y)v(X) - u(X)v(Y)\}.\end{aligned}$$

Taking $Y = U,\ X = \xi$ and $Y = U,\ X = V$ to this by turns, we obtain

$$f_1 - f_2 = \alpha^2 - \beta^2, \qquad \alpha\beta = 0.$$

Applying $\bar{\nabla}_X$ to $\theta(\zeta) = 1$ and using (2.3), we obtain

$$(\bar{\nabla}_X\theta)(\zeta) = 0. \tag{5.8}$$

Applying $\bar{\nabla}_X$ to $\eta(Y) = \bar{g}(Y,N)$ and using (2.4) and (2.5), we have

$$(\nabla_X\eta)(Y) = -g(A_N X,\, Y) + \tau(X)\eta(Y). \tag{5.9}$$

Applying $\nabla_Y$ to (3.8) and using (3.6), (3.8), (3.15) and (5.9), we get

$$\begin{aligned}(\nabla_X C)(Y,\zeta) &= -(X\alpha)v(Y) + (X\beta)\eta(Y)\\ &\quad - \alpha\{v(Y)\tau(X) + w(Y)\rho(X)\\ &\qquad - g(A_N X, FY) - g(A_N Y, FX)\\ &\qquad - \alpha\theta(Y)\eta(X) + \theta(X)v(Y) - \theta(Y)v(X)\}\\ &\quad + (\beta+1)\{\tau(X)\eta(Y) - g(A_N X, Y) - g(A_N Y, X)\\ &\qquad + (\beta+1)\theta(X)\eta(Y)\}.\end{aligned}$$

Substituting this and (3.8) into (5.6) with $PZ = \zeta$ and using (5.8), we get

$$\begin{aligned}&-(X\alpha)v(Y) + (Y\alpha)v(X) + (X\beta)\eta(Y) - (Y\beta)\eta(X)\\ &= (f_1 - f_3 - \alpha^2 + \beta^2)\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}.\end{aligned}$$

Taking $Y = \zeta,\ X = \xi$ and $Y = U,\ X = V$ to this by turns, we obtain

$$f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta, \qquad U\alpha = 0.$$

Applying $\nabla_Y$ to $(3.7)_1$ and using (3.6), $(3.7)_1$ and (3.14), we have

$$\begin{aligned}(\nabla_X B)(Y,\zeta) &= -(X\alpha)u(Y) - (\beta+1)B(X,Y)\\ &\quad + \alpha\{u(Y)\tau(X) + w(Y)\phi(X)\\ &\qquad + \theta(Y)u(X) - \theta(X)u(Y)\\ &\qquad + B(X,FY) + B(Y,FX)\}.\end{aligned}$$

Substituting this equation and $(3.7)_1$ into (5.5) with $Z=\zeta$, we obtain

$$(X\alpha)u(Y) = (Y\alpha)u(X).$$

Taking $Y=U$ to this, we get $X\alpha = 0$. Thus $\alpha$ is a constant on $M$. □

**Theorem 5.2.** *Let $M$ be a Lie recurrent lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ with a semi-symmetric metric connection. Then $\bar{M}(f_1,f_2,f_3)$ is a space form such that*

$$f_1 = -\beta^2, \qquad f_2 = 0, \qquad f_3 = \zeta\beta.$$

*Proof.* As $\tau(FX)=0$, from (4.12) we obtain

$$B(Y,U) = 0. \tag{5.10}$$

Applying $\nabla_X$ to this and using $(3.7)_1$, (3.10), (5.10) and $\alpha=0$, we have

$$(\nabla_X B)(Y,U) = -B(Y,F(A_N X)) - \rho(X)B(Y,W).$$

Substituting the last two equations into (5.5), we have

$$\begin{aligned}&B(X,F(A_N Y) - B(Y,F(A_N X) + \rho(Y)B(X,W)\\ &- \rho(X)B(Y,W) + \phi(X)D(Y,U) - \phi(Y)D(X,U)\\ &= \frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X,JY)\}.\end{aligned}$$

Taking $X=\xi$ and $Y=U$ to this and using $(2.11)_{1,2}$, $(4.10)_{3,5}$ and (5.10), we get $f_2=0$. Thus $f_1=-\beta^2$ and $f_3=\zeta\beta$ by Theorem 5.1. □

**Definition.** (1) A screen distribution $S(TM)$ is said to be *totally umbilical* [6] if there exists a smooth function $\gamma$ on a neighborhood $\mathcal{U}$ such that

$$C(X,PY) = \gamma g(X,PY). \tag{5.11}$$

In case $\gamma=0$, we say that $S(TM)$ is *totally geodesic* in $M$.

(2) A half lightlike submanifold $M$ is said to be *screen conformal* [6] if there exists a non-vanishing smooth function $\varphi$ on $\mathcal{U}$ such that

$$C(X,PY) = \varphi B(X,Y). \tag{5.12}$$

**Theorem 5.3.** *Let $M$ be a half lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ with a semi-symmetric metric connection. If one of the following five conditions is satisfied;*

(1) *$M$ is recurrent,*
(2) *$M$ is screen conformal,*
(3) *$S(TM)$ is totally umbilical,*
(4) *$U$ is parallel with respect to the induced connection $\nabla$, and*
(5) *$V$ is parallel with respect to the induced connection $\nabla$,*

*then $\bar{M}(f_1,f_2,f_3)$ is an indefinite Kenmotsu space form such that*

$$\alpha = 0, \quad \beta = -1; \qquad f_1 = -1, \quad f_2 = f_3 = 0.$$

*Proof.* Applying $\bar{\nabla}_X$ to $\theta(U) = 0$ and using (2.4) and (3.10), we obtain

$$(\bar{\nabla}_X\theta)(U) = \alpha\eta(X) + (\beta + 1)v(X). \tag{5.13}$$

(1) As $M$ is recurrent, by Theorem 4.1, we show that $\alpha = 0$, $\beta = -1$ and $\rho = 0$. By directed calculation from $(4.5)_1$, we obtain

$$R(X, Y)U = 2d\tau(X, Y)U. \tag{5.14}$$

On the other hand, since $\alpha = 0$ and $\beta = -1$, we have $\bar{\nabla}_X\zeta = 0$ by (2.3) and $f_1 + 1 = f_2 = f_3$ by Theorem 5.1. Comparing the tangential components of the right and left terms of (5.2) and using (5.1) and (5.3), we obtain

$$\begin{aligned} R(X, Y)Z = {} & B(Y, Z)A_{_N}X - B(X, Z)A_{_N}Y \\ & + D(Y, Z)A_{_L}X - D(X, Z)A_{_L}Y \\ & + (\bar{\nabla}_X\theta)(Z)Y - (\bar{\nabla}_Y\theta)(Z)X \\ & + (f_1 + 1)\{g(Y, Z)X - g(X, Z)Y\} \\ & + f_2\{\bar{g}(X, JZ)FY - \bar{g}(Y, JZ)FX + 2\bar{g}(X, JY)FZ\} \\ & + f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\ & \qquad + \bar{g}(X, Z)\theta(Y)\zeta - \bar{g}(Y, Z)\theta(X)\zeta\}, \end{aligned}$$

Replacing $Z$ by $U$ to this and using (5.13) and (5.14), we get

$$\begin{aligned} 2d\tau(X, Y)U = {} & B(Y, U)A_{_N}X - B(X, U)A_{_N}Y \\ & + D(Y, U)A_{_L}X - D(X, U)A_{_L}Y \\ & + (f_1 + 1)\{v(Y)X - v(X)Y\} \\ & + f_2\{\eta(X)FY - \eta(Y)FX\} \\ & + f_3\{v(X)\theta(Y) - v(Y)\theta(X)\}\zeta. \end{aligned}$$

Taking the scalar product with $N$ to this and using $(2.12)_{1,2}$, we get

$$2f_2\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

It follow that $f_2 = 0$. Thus $f_1 = -1$ and $f_2 = f_3 = 0$.

(2) Taking $Y = \zeta$ to (5.12) and using $(3.7)_1$ and (3.8), we obtain

$$\alpha v(X) - (\beta + 1)\eta(X) = \alpha\varphi u(X).$$

Taking $X = V$ and $X = \xi$ by turns, we have $\alpha = 0$ and $\beta = -1$. Thus $\bar{M}$ is an indefinite Kenmotsu manifold, and $f_1 + 1 = f_2 = f_3$ by Theorem 5.1.

Let $\mu = U - \varphi V$. Then $J\mu = N - \varphi\xi$. Using (3.9) and (5.12), we obtain

$$B(X, \mu) = 0, \qquad D(X, \mu) = 0. \tag{5.15}$$

Also, using (5.7), (5.13) and the fact that $\mu = U - \varphi V$, we get

$$(\bar{\nabla}_X\theta)(\mu) = 0. \tag{5.16}$$

Applying $\nabla_X$ to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this equation and (5.12) into (5.6) and using (5.5), we have

$$\begin{aligned}
&\{X\varphi - 2\varphi\tau(X)\}B(Y,PZ) - \{Y\varphi - 2\varphi\tau(Y)\}B(X,PZ)\\
&- \{\rho(X) + \varphi\phi(X)\}D(Y,PZ) + \{\rho(Y) + \varphi\phi(Y)\}D(X,PZ)\\
&- \{(\bar\nabla_X\theta)(PZ) - g(X,PZ)\}\eta(Y)\\
&+ \{(\bar\nabla_Y\theta)(PZ) - g(Y,PZ)\}\eta(X)\\
&= f_1\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y)\}\\
&+ f_2\{[v(Y) - \varphi u(Y)]\bar g(X,JPZ) - [v(X) - \varphi u(X)]\bar g(Y,JPZ)\\
&\qquad +2[v(PZ) - \varphi u(PZ)]\bar g(X,JY)\}\\
&+ f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ).
\end{aligned}$$

Replacing $PZ$ by $\mu$ to this and using (5.15) and (5.16), we obtain

$$f_2\{v(Y)\eta(X) - v(X)\eta(Y) - \varphi[u(Y)\eta(X) - u(X)\eta(Y)] - 2\varphi\bar g(X,JY)\} = 0.$$

Taking $X = \xi$ and $Y = V$, we get $f_2 = 0$. Thus $f_1 = -1$ and $f_2 = f_3 = 0$.

(3) If $S(TM)$ is totally umbilical, then, from (3.8) and (5.11), we have

$$\gamma\theta(X) = -\alpha v(X) + (\beta + 1)\eta(X).$$

Taking $X = \zeta$, $X = V$ and $X = \xi$ by turns, we have $\gamma = 0$, $\alpha = 0$ and $\beta = -1$ respectively. As $\gamma = 0$, $S(TM)$ is totally geodesic in $M$. As $\alpha = 0$ and $\beta = -1$, $\bar M$ is an indefinite Kenmotsu manifold, and $f_1 + 1 = f_2 = f_3$ by Theorem 5.1. As $C = 0$, taking $PZ = U$ to (5.6) and using $(3.9)_2$: $D(X,U) = 0$, (5.13) and the fact that $f_1 + 1 = f_2$, we obtain

$$2f_2\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = \xi$ and $Y = V$, we get $f_2 = 0$. Thus $f_1 = -1$ and $f_2 = f_3 = 0$.

(4) If $U$ is parallel with respect to $\nabla$, then we have (3.16) and (3.17), From $(3.16)_{1,2}$, we shown that $\bar M$ is an indefinite cosymplectic manifold. Also, the functions $f_1$, $f_2$ and $f_3$ satisfy $f_1 + 1 = f_2 = f_3$ by Theorem 5.1. Taking the scalar product with $U$ to (3.17), we obtain

$$C(X,U) = 0.$$

Applying $\nabla_X$ to $C(Y,U) = 0$ and using the fact that $\nabla_X U = 0$, we have

$$(\nabla_X C)(Y,U) = 0.$$

Substituting the last two equations into (5.6) with $PZ = U$ and using (5.13) and the fact that $f_1 + 1 = f_2$, we obtain

$$2f_2\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = \xi$ and $Y = V$, we get $f_2 = 0$. Thus $f_1 = -1$ and $f_2 = f_3 = 0$.

(5) If $V$ is parallel with respect $\nabla$, then we have (3.18) and (3.19). From $(3.19)_{1,2}$, we shown that $\bar M$ is an indefinite cosymplectic manifold. Also, the functions $f_1$, $f_2$ and $f_3$ satisfy $f_1 + 1 = f_2 = f_3$ by Theorem 5.1.

As $B(X,U) = 0$, from $(3.9)_1$ we obtain

$$C(X,V) = 0. \tag{5.17}$$

By using $(3.9)_{1,3}$ and (5.17), we see that

$$D(U,V) = B(U,W) = C(W,V) = 0. \tag{5.18}$$

Applying $\nabla_X$ to $C(Y,V)=0$ and using the fact that $\nabla_X V=0$, we have

$$(\nabla_X C)(Y,V)=0.$$

Substituting the last equation and (5.17) into (5.6) with $PZ=V$ and using (5.7) and the fact that $f_1+1=f_2$, we have

$$\begin{aligned}&-\rho(X)D(Y,V)+\rho(Y)D(X,V)\}\\&= f_2\{u(Y)\eta(X)-u(X)\eta(Y)+2\bar{g}(X,JY)\}.\end{aligned}$$

Taking $X=\xi$ and $Y=U$ and using $(2.11)_2$, (5.18) and the fact that $\phi=0$, we get $f_2=0$. Thus $f_1=-1$ and $f_2=f_3=0$. □

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